Asa Scherer
April 5, 2004
Hon 213
Professor Bryan Smith

Formalism and Intuitionism… Similar?
Assumptions as Requirements in Mathematics

Communication requires common assumptions on the part of the speaker and the listener. For example, if I were to tell someone who believed as an underlying truth that God exists that God didn't exist, they would know that I was wrong, and no matter what arguments I used, they still wouldn't believe me. (The opposite would work equally well in this case.) Of course, the existence of God is such a controversial example that many people have completely different views on the subject, making logical arguments on this topic very difficult. For a positive example, in order for a person to communicate a logical argument backed by facts, both people must assume that these facts are, indeed, true, or else the listener may be disinclined to agree with the argument.

But what about mathematics? What implicit assumptions are made while communicating proofs? Formalists and intuitionists present two seemingly different answers to this question. Formalists say that once we assume the axioms and symbols within a given system, we advance mathematics through logical proof using these symbols. Intuitionists would say that every human being inherently understands certain fundamental mathematical concepts and that proofs must emanate from and build upon these archetypes. I say that these assumptions are *seemingly* different because each branch requires underlying assumptions very similar to the other in order to effectively communicate a proof.

First of all, I'll break down the readily apparent assumptions necessary for communication in each branch. In Formalism, these assumptions are stated quite blatantly as the axioms of the given mathematical system they are working in, paired with the tools of logic and symbols to represent the undefined terms within the axioms. These elements are designed very carefully so that no assumptions are made about the undefined terms themselves in order to avoid infinite regress of definitions.

In Intuitionism, on the other hand, the assumptions of the system are much more philosophical, and less practical. These assumptions deal with the idea that every human being has the capacity for mathematical reasoning, and the way people reason is consistent from person to person. (This should not be confused with classical logic, as Intuitionists reject the law of excluded middle on the grounds that this rule cannot be constructed based on mathematical reasoning.) For example, one Intuitionist axiom is that all humans possess the concept of "one," and all the integers can be derived from this idea. Thus, according to the Intuitionist, math should be easily understood and recognized by any normal human being.

It should be noted that, while assumptions are certainly necessary in communication, they can also be seen as a negative thing, something that must be eliminated in order to gain access to pure Truth. An Intuitionist attacking a Formalist might argue that axiomatic systems are too arbitrary and artificial to really reflect anything meaningful, while a Formalist would counter that assuming that every person reasons the same is unrealistic; perhaps people reason differently, according to how they are taught to reason, and creating self-contained systems is the only way to consistently extract truths from mathematics, since everyone is told to assume the same things. Of course, each discipline is completely valid in its own way, and this is largely because of their respective purposes.

Classical mathematics, though constructed very logically and systematically, was also created over the course of thousands of years by a large variety of people. And though these mathematics have shown to be useful in many applications such as physics and probability, there still exist contradictions that trouble anyone involved enough in mathematics to care. Formalism, still wanting to keep classical mathematics intact, seeks to "formalize" classical mathematics through the creation of very carefully worded axioms and logical language, so that it is impossible for contradictions to arise. Thus, creating arbitrary axioms is *necessary* for Formalism's purpose.

Intuitionists have no intention of proving all of classical mathematics, as they believe that contradictions arose because of non-constructive arguments. In fact, they start math completely from scratch, beginning with the philosophical axioms about innate human capacity for mathematical reasoning, and simply seeing where this reasoning takes them. They frequently have proven aspects of classical mathematics, but when they cannot do so because of certain concepts that they feel are not constructed from the basic human reasoning abilities, they do not pursue it. (Interestingly enough, though, as A. Heyting points out in his book *Intuitionism: An Introduction*, Intuitionists also seek to drive the metaphysical from mathematical construction so there is no possibility of contradictions arising from "mind games" that have, according to the intuitionist, no basis in reality.) (Heyting, 2) So Intuitionism seeks to develop a sort of ground-up mathematics, one that is firmly planted in the philosophical concept of human reasoning.

The legitimacy of this branch of mathematics is really dependent upon the philosophy of the observer: is a person born with the Tabula Rasa of the Sophists, or the already-present knowledge of Socrates? Intuitionists, like Socrates, would say that people are born with innate "knowledge," or at least ability to process information in a specific way, while people who disagree with Intuitionism might argue that the human mind is largely a Tabula Rasa, or "blank slate" that is only filled through experience, and therefore each person could have different ideas about mathematics and logic depending on their experience. Thus Formalism defends its truth value by definition (or re-definition, if necessary), and Intuitionism by a specific philosophy. It would seem that Formalism has the upper hand in communicating its ideas.

However, let us examine more closely the underlying assumptions of Formalism. On the surface, they are, indeed, merely axioms created to support a certain area of mathematics. But where do *those* axioms come from, and what makes them valid? And, more importantly, *why* are they being created? Well, the gist of the axioms comes from the mathematicians who pioneered that particular branch of mathematics. And they were created usually to describe the natural world (most likely via physics.) In this point lies a hidden assumption: that math *is* fundamentally meaningful and inherent to some aspect of reality. True Formalists may argue with this, saying that math is simply a meaningless series of numbers and symbols, and the Formalistic proof of something is true simply because the axioms are defined to be true, yet these symbols can be used to very accurately represent the world we live in. So ultimately, the purpose of Formalism itself is to clarify and purify this *meaningful* representation by detaching themselves from its meaning.

Now, this is not to say that to communicate Formalism requires more basic assumptions than Intuitionism. Indeed, to prove something Formalistically, all one needs to assume is that the axioms hold, and then they just follow the logic from there. Of course, it technically doesn't need to be assumed that the logic works, since it is defined to work in a very organized way. But what is it defined with? This is where things get a little hazy. The logical manner in which symbols can be arranged is defined with the human language, and understanding the definition requires understanding the *concepts* "True," "False," and "Implication," among others. Thus, to understand the logical flow of a proof, one must assume that logic holds.

This aspect of Formalism is very similar to Intuitionism, since all mathematics uses some form of logic to deduce truth. Intuitionism, though, with its assumption of human reasoning, does not attempt to *formally* (that is, with similar devices to Formalism) define logic, but rather creates logical tools with respect to natural language and inherent logical reasoning. For example, Paul Lorenzen, in his book *Constructive Philosophy*, describes one logical tool through the example of classifying different types of living beings into different sub-categories, a fairly basic, primitive example. (Lorenzen, 9) This is not to say, though, that Intuitionistic logical concepts are not well-defined. Lorenzen goes to great length to show each concept and how it relates to our pre-established natural language so that there can be little to no ambiguity on what he means. In this sense, when logical devices are being defined, Intuitionism seems very similar to Formalism in simply choosing an assumption that makes the concept "work" within the system, as there could be multiple interpretations of a given logical argument made in the English language. So, in Intuitionism, one must assume that all the logical concepts explained are inherent to human nature.

In this respect, Formalism and Intuitionism are fairly similar. In order to understand a proof as true or false, one must first know the implications of "true" and “false” (as well as “implies,” etc.) That is, any proof in math requires common logical ground on the part of the prover and of the observer. Of course, logic is so ingrained in us that it is hard to imagine not assuming that its rules hold. And we can’t define logic logically, since that would basically be saying that “true is true.” Though Formalists might say that this circular definition is circumvented by the axiomatic system of logic, axiomatic systems themselves rely on basic logical concepts, so logic must ultimately still be assumed. So this leaves us with two interpretations: that people recognize logical arguments because they are taught to do so, or because it is a fundamental ability inherent to all human beings. The philosophical implications of these choices are obviously different, and once again factor in with whether an individual agrees with Intuitionism’s axioms or not, but the point remains that logic must be assumed to hold in both Formal and Intuitionistic proofs.

So, though they disagree bitterly on many topics of mathematics, Formalism and Intuitionism are both actually quite similar to each other in terms of assumptions required for communication. As Heyting says, “I see the difference between formalists and intuitionists mainly as one of tastes.” (Heyting, 4) Luckily, I find both fairly tasty.

Resources

Heyting, A. *Intuitionism: An Introduction*. North-Holland Publishing Company, Amsterdam, 1971.

Lorenzen, Paul. *Constructive Philosophy*. The University of Massachusetts Press, 1987. Translated by Karl Richard Pavlovic.